NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Procedural Fluency in Mathematics

A Position of the National Council of Teachers of Mathematics

Question

What is procedural fluency, and how do we help students develop it?

NCTM Position

Procedural fluency is a critical component of mathematical proficiency. Procedural fluency is the ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts; to build or modify procedures from other procedures; and to recognize when one strategy or procedure is more appropriate to apply than another. To develop procedural fluency, students need experience in integrating concepts and procedures and building on familiar procedures as they create their own informal strategies and procedures. Students need opportunities to justify both informal strategies and commonly used procedures mathematically, to support and justify their choices of appropriate procedures, and to strengthen their understanding and skill through distributed practice.

Procedural fluency is more than memorizing facts or procedures, and it is more than understanding and being able to use one procedure for a given situation. Procedural fluency builds on a foundation of conceptual understanding, strategic reasoning, and problem solving (NGA Center & CCSSO, 2010; NCTM, 2000, 2014). Research suggests that once students have memorized and practiced procedures that they do not understand, they have less motivation to understand their meaning or the reasoning behind them (Hiebert, 1999). Therefore, the development of students' conceptual understanding of procedures should precede and coincide with instruction on procedures. Although conceptual knowledge is an essential foundation, procedural knowledge is important in its own right. All students need to have a deep and flexible knowledge of a variety of procedures, along with an ability to make critical judgments about which procedures or strategies are appropriate for use in particular situations (NRC, 2001, 2005, 2012; Star, 2005).

In computation, procedural fluency supports students' analysis of their own and others' calculation methods, such as written procedures and mental methods for the four arithmetic operations, as well as their own and others' use of tools like calculators, computers, and manipulative materials (NRC, 2001). Procedural fluency extends students' computational fluency and applies in all strands of mathematics. For example, in algebra, students develop general equation-solving procedures that apply to classes of problems and select efficient procedures to use in solving specific problems. In geometry, procedural fluency might be evident in students' ability to apply and analyze a series of geometric transformations or in their ability to perform the steps in the measurement process accurately and efficiently.

Procedural fluency builds from an initial exploration and discussion of number concepts to using informal reasoning strategies and the properties of operations to develop general methods for solving problems (NCTM, 2014). Effective teaching practices provide experiences that help students to connect procedures with the underlying concepts and provide students with opportunities to rehearse or practice strategies and to justify their procedures. Practice should be brief, engaging, purposeful, and distributed (Rohrer, 2009). Too much practice too soon can be ineffective or lead to math anxiety (Isaacs & Carroll, 1999). Analyzing students' procedures often reveals insights and misunderstandings that help teachers in planning next steps in instruction. In the same way, worked examples can serve as a valuable instructional tool, permitting teachers to understand how students analyze why procedures work or don't work and consider what procedure might be most appropriate in a given situation (Booth, Lange, Koedinger, & Newton, 2013).

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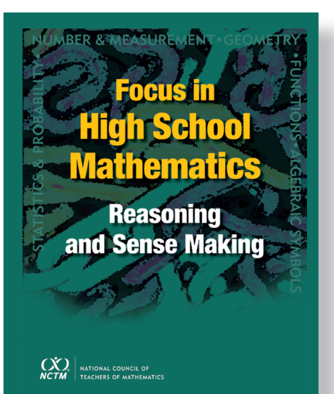
Executive Summary

Focus in High School Mathematics: Reasoning and Sense Making

Building on three decades of advocacy for Standards-based mathematics learning of the highest quality for all students, a new publication of the National Council of Teachers of Mathematics recommends that all high school mathematics programs focus on reasoning and sense making. In recent years, a number of documents have provided detailed analyses of the topics that should be addressed in each course of high school mathematics (see, for example, American Diploma Project [2004]; College Board [2006, 2007]; ACT [2007]; Achieve [2007a, 2007b]). NCTM's *Focus in High School Mathematics: Reasoning and Sense Making* offers a different perspective, proposing curricular emphases and instructional approaches that make reasoning and sense making foundational to the content that is taught and learned in high school.

A high school mathematics program based on reasoning and sense making will prepare students for citizenship, the workplace, and further study.

High school students face major challenges in their mathematics preparation. U.S. students lag in basic mathematical literacy—the knowledge and skills that prepare them to apply mathematics in a variety of contexts, including their future lives as responsible citizens (see, for example, the Programme for International Student Assessment [2007]). They are not prepared to face the economic and workforce challenges of an increasingly global, technological society. This inadequate preparation is contributing to the decline of U.S. leadership in many technical fields (see Tapping America's Potential [2008]). *Focus in High School Mathematics: Reasoning and Sense Making* argues that focusing on reasoning and sense making in the context of strong mathematical content will help high school students meet these challenges.



Reasoning involves drawing conclusions on the basis of evidence or assumptions. Although reasoning is an important part of all disciplines, it plays a special role in mathematics. In addition to formal reasoning or proof, reasoning in mathematics often begins with explorations, conjectures, or false starts. As students progress through the high school years, they should develop increasingly sophisticated standards for explanations. Sense making involves developing an understanding of a situation, context, or concept by connecting it with existing knowledge. Reasoning and sense making are closely intertwined and interdependent.

Reasoning and sense making are the foundations for the processes of mathematics—problem solving, reasoning and proof, connections, communication, and representation (see NCTM [2000]). Moreover, reasoning and sense making help students develop connections between new learning and their existing knowledge, increasing their likelihood of understanding and retaining the new information. (As this volume uses the term *reasoning*, mathematical reasoning encompasses statistical reasoning.)

Reasoning and sense making should be a part of the mathematics classroom every day.

Focus in High School Mathematics: Reasoning and Sense Making describes "reasoning habits," which are productive ways of thinking that should become customary in the processes of mathematical inquiry and sense making. In addition to "covering" mathematical topics, high school mathematics programs must give attention to developing these reasoning habits on a continuing basis—not as a set of new topics to be taught but as an integral part of the curriculum. The publication offers a list of sample reasoning habits, which it emphasizes are not experienced in isolation or sequentially. To help their students progress to higher levels of reasoning, teachers must judiciously select tasks that require them to figure things out for themselves and ask probing questions. Both teachers and students should ask and answer such questions as "What's going on here?" and "Why do you think that?"

Reasoning and sense making are inherent in developing the components of mathematical competence (Kilpatrick, Swafford, and Findell 2001). Conceptual understanding is interrelated with sense making as defined in this volume. Procedural fluency includes not only knowing how to carry out procedures, but also understanding why they work, how they might be used, and how their results should be interpreted, all of which are grounded in reasoning and sense making. Other aspects of mathematical competence directly draw on reasoning and sense-making skills, including the ability to formulate, represent, and solve mathematical problems and the capacity for logical thought and explanation. Finally, students who view mathematics as a reasoning and sense-making enterprise may be more likely to develop a productive disposition toward mathematics.

Mathematics should help students understand and operate in the physical and social worlds. They should be able to connect mathematics with a real-world situation through the use of mathematical models. The connections between mathematics and real-world problems developed in mathematical modeling add value to, and provide incentive and context for, studying mathematical topics.

Technology is an integral part of the world in which students live, and high school mathematics classrooms must reflect that reality. Technology can advance the goals of reasoning and sense making—facilitating students' searches for patterns, reducing the load of burdensome calculations so that they can focus on thinking strategically, and providing them with multiple ways of representing a mathematical situation. However, the use of technology should not be allowed to overshadow the development of procedural proficiency. Students who have opportunities to reflect on how to use technological tools effectively will be less likely to use them as a crutch.

Reasoning and sense making are integral to the experiences of all students across the high school mathematics curriculum.

Reasoning and sense making should be pervasive in all areas of the high school mathematics curriculum. Although formal reasoning is often emphasized in geometry, students are less likely to experience reasoning in other areas of the curriculum, such as algebra. When reasoning and sense making are infused everywhere in the curriculum, they allow students to discover coherence across the domains of mathematics and help them see how new concepts connect with existing knowledge. Although "reasoning" should not be viewed as a set of new topics but rather a stance toward mathematics learning, a focus on reasoning and sense making will inevitably require instructional time. However, developing strong reasoning habits may yield compensating efficiencies. Students who make reasoned connections with existing knowledge may be more likely to retain what they have learned in previous courses, thus reducing the need for reteaching. Furthermore, instruction that emphasizes underlying connections among ideas may provide coherence that allows for streamlining the curriculum and eliminating lists of particular skills that teachers must help students to master.

Focus in High School Mathematics highlights reasoning opportunities in five specific content areas of the high school mathematics curriculum:

- Reasoning with Numbers and Measurements
- Reasoning with Algebraic Symbols
- Reasoning with Functions
- Reasoning with Geometry
- Reasoning with Statistics and Probability

Within each content area, the publication identifies a number of key elements that provide a broad structure for considering possible ways of focusing on reasoning and sense making. These key elements are not intended to be an exhaustive list of specific topics to be addressed but rather a lens through which to view the potential of high school programs for promoting mathematical reasoning and sense making. A separate chapter on each content area focuses on how reasoning and sense making can be promoted within the key elements of that area. The chapters also include a series of examples intended to provide idealized illustrations of how reasoning and sense making might develop.

The task of creating a curriculum that realizes the goals of this document will be challenging. Although such a curriculum must address important content, its creation requires much more than developing lists of topics to be taught in particular courses. Moreover, students must have experiences with reasoning and sense making within a broad curriculum that meets a wide range of their future needs, preparing them for future success as citizens and in the workplace, as well as for careers in mathematics and science.

Mathematical reasoning and sense making must be evident in the mathematical experiences of all students.

Essential to realizing the vision for high school mathematics outlined in this publication is ensuring that all students—no matter their mathematical background or the mathematics class in which they are enrolled—have full access to opportunities for reasoning and sense making in their mathematics classes. *Focus in High School Mathematics: Reasoning and Sense Making* provides high school teachers, administrators, and staff with some considerations for ensuring that their schools are enacting equitable learning for all their students. In particular, *Focus in High School Mathematics* communicates the message that high schools can monitor equity by attending to phenomena that potentially pose barriers to, or have a significant impact on, the opportunities for engaging every student in the activities of reasoning and sense making. These phenomena include the following:

- ***Courses.*** It is very important that high schools look critically at their practices involving tracking or grouping of students by ability. The courses that students take have an impact on the opportunities that they have for reasoning and sense making. Students in all levels of mathematics—from prealgebra to calculus and from low-track to Advanced Placement—must have mathematical experiences that offer rich opportunities to build reasoning habits as well as to make sense of what they are doing mathematically.
- ***Students' demographics.*** Mathematics educators continue to be concerned about discrepancies in achievement among demographic groups on the basis of race, ethnicity, socioeconomic status, and other variables. Students from some groups receive fewer opportunities for reasoning and sense making than students in other groups. As a result, it is important that high schools do everything they can to promote success among all students—for example, encouraging enrollment by students from all demographic groups in advanced mathematics courses. Also, providing students with opportunities to see that mathematics is important for their lives and future career is a must for high schools.
- ***Expectations, beliefs, and biases.*** The expectations, beliefs, and biases of others can significantly affect the mathematical opportunities provided for students. Building on *Principles and Standards for School Mathematics* (NCTM 2000), *Focus in High School Mathematics: Reasoning and Sense Making* emphasizes the need for teachers, administrators, and school staff to hold high expectations for all students. Teachers' beliefs about students' mathematical capabilities can have serious implications for the opportunities that students are afforded in high school mathematics. Teachers must believe that students will benefit from and can engage in reasoning and sense making, and they must work to help students succeed in this endeavor.

Curriculum, instruction, and assessment form a coherent whole to support reasoning and sense making.

To achieve the vision of reasoning and sense making as the focus of high school students' mathematical experience, all the components of the educational system—curriculum, instruction, and assessment—must work together and be designed to support students' reasoning and sense making. A coherent and cohesive mathematics program requires strong alignment of these three elements.

This publication's recommendations, in combination with the more detailed content recommendations in *Principles and Standards for School Mathematics*, provide a critical filter for examining any curriculum arrangement to ensure the achievement of the goals of high school mathematics. Although curriculum is undeniably crucial to reaching the goal set out in this publication, it cannot stand alone. Mathematical instruction and a classroom environment promoting and valuing students' reasoning and sense making are essential as well. Teachers must select worthwhile tasks that engage students in reasoning and sense making.

As students move through mathematics from prekindergarten through college, coherence in curriculum and instruction is crucial to their success. Too often, as students progress, they fall victim to differing mathematical expectations. To achieve the goal of curricular coherence, an open dialogue is essential among prekindergarten through grade 8 teachers, secondary school mathematics teachers, mathematics teacher educators, and mathematics and statistics faculty in higher education, as well as others in client disciplines, to ensure continuing support and development of students' mathematical abilities.

Schools, parents, policymakers, and others need to see evidence that the development of reasoning and sense-making abilities is a shared goal at all levels of mathematics teaching, including elementary school mathematics, high school mathematics, and the undergraduate curriculum. The time is right to build strong partnerships and recognize the benefits of a mathematics curriculum that focuses on reasoning and sense making from prekindergarten through grade 16.

Finally, realizing any goal for students' learning involves assessment. Schools are currently under tremendous pressure to demonstrate success as measured by high-stakes tests. It is important to assess what we value. Assessments that support the goals of this publication will probe students' development of mathematical reasoning and sense making and contribute to students' progress in mathematics. This endeavor is essential for at least two reasons. First, we will not be able to determine whether we are meeting our goals if we do not measure our progress. Second, high-stakes testing that concentrates primarily on procedural skills *without* assessing reasoning and sense making sends a message that is contrary to the vision of *Focus in High School Mathematics* and can adversely influence instruction and learning. Assessment that focuses primarily on students' abilities to do algebraic manipulations, apply geometric formulas, and perform basic statistical computations will lead students to believe that reasoning and sense making are not important.

Formative assessment—which involves providing students with learning activities and, on the basis of feedback from those activities, adjusting teaching to meet the students' needs—is important in helping teachers ensure that their students' reasoning and sense making are progressing. Formative assessments rely on a variety of tools, including teacher observations, classroom discussions, student journals, student presentations, homework, and in-class tasks, as well as tests and quizzes that ask students to explain their thinking.

All stakeholders must work together to ensure that reasoning and sense making are the focus of high school mathematics programs.

Focus in High School Mathematics: Reasoning and Sense Making presents an ambitious vision for the improvement of high school mathematics. Its refocusing of the high school curriculum on reasoning and sense making is not a minor tweaking but a substantial rethinking of the high school mathematics curriculum and requires the engagement of all involved in high school mathematics.

Significant effort will be needed to realign the curriculum. To develop new understanding, teachers will need long-term professional development and support, including opportunities for reflection on their practice and guidance in improving it. Students must be offered the resources needed to prepare them for our rapidly changing world and must recognize that studying mathematics in high school is important to their future careers. Families should understand which mathematics classes are important for their students to take and should help them develop good study and homework habits. Teachers must believe in—and communicate their conviction about—the importance of reasoning and sense making for every student in every mathematics topic.

Together, school districts, schools, departments, and teachers must ensure that high school students are exposed to a high-quality mathematics curriculum that promotes reasoning and sense making. State and local assessments policies should emphasize the need for and importance of items that examine students' abilities to reason and make sense of mathematical situations. In addition, policymakers must secure adequate resources to assist schools and districts in efforts to implement an effective curriculum based on reasoning and sense making.

This publication provides a framework for considering necessary changes to the high school mathematics curriculum and how those changes might be made. However, many issues beyond those addressed in this publication remain to be answered. Future publications, including an initial set of topic books that set forth additional guidance in particular content areas, will offer resources that build on this framework. Although NCTM is taking a leadership role, all stakeholders must join forces and work together in meaningful ways to ensure that the story of missed opportunities to improve high school mathematics across the United States does not continue, to be told five years from now, let alone in three decades. We simply cannot afford to wait any longer to address the large-scale changes that are needed. The success of our students and of our nation depends on it.

(August 14, 2009)

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PROMOTING EQUITY THROUGH REASONING

By Mary F. Mueller and Carolyn A. Maher

Learn how five characteristics of tasks and learning environments led these sixth graders to successful problem solving using direct and indirect reasoning to justify their solutions, make their justifications public, and respond to mathematical arguments.

Many educators share the vision of the Equity Principle—teachers holding high expectations for all students (NCTM 2000). However, according to National Assessment of Educational Progress (NAEP) reports, minority students continue to lag behind white students in mathematics achievement (Strutchens and Silver 2000). For example, on the 2000 NAEP mathematics assessment, 34 percent of white fourth graders scored at or above "proficient" compared to 5 percent of black students and 10 percent of Hispanic students (Braswell et al. 2001). Furthermore, the discrepancies are more pronounced on the extended, constructed-response items, which measure students' problem-solving and critical-thinking abilities (Kloosterman and Lester 2004). Several factors contribute to the failure of minority students to build meaningful mathematics learning in schools:

1. Low expectations for students' success in building conceptual mathematical knowledge
2. Classroom environments in which students are insufficiently challenged with thoughtful and engaging mathematical activities

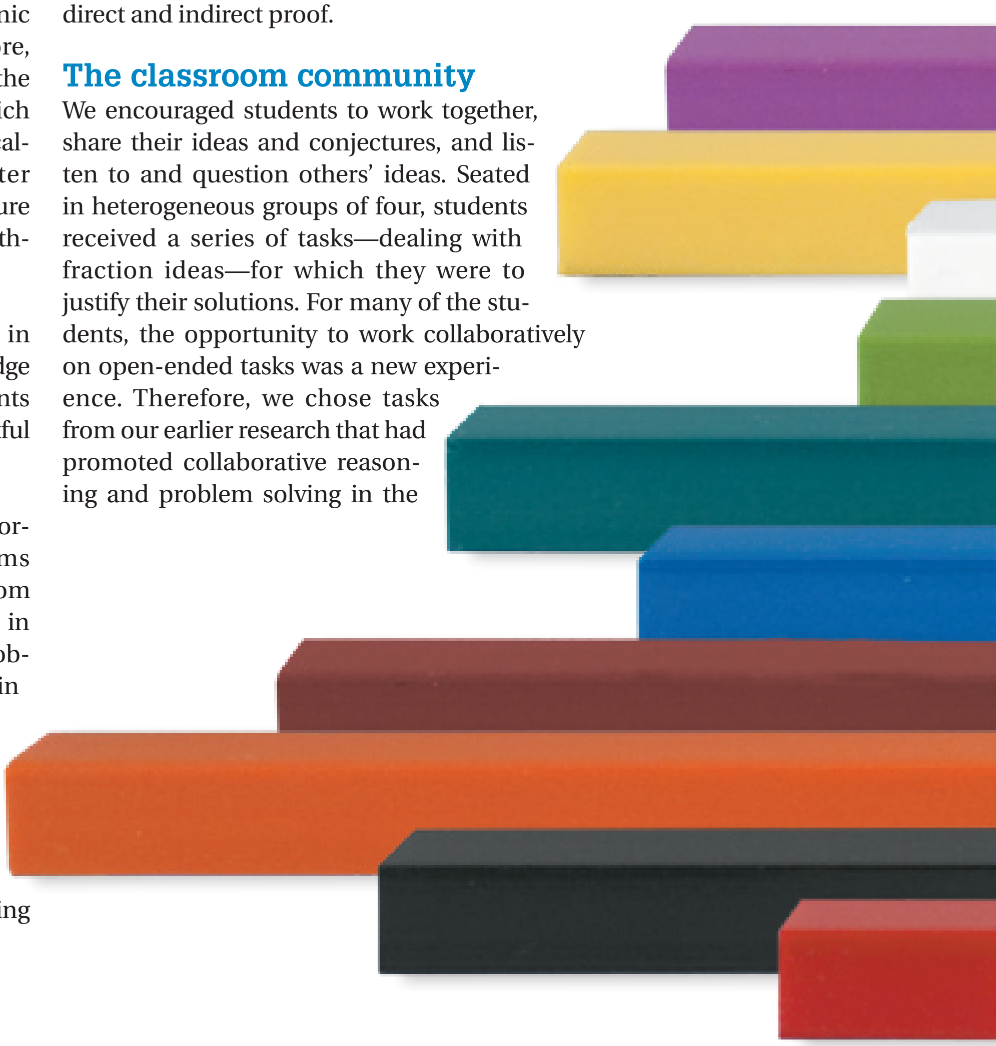
On the other hand, recognizing the importance of equitable practices in classrooms suggests optimism for achieving a classroom community where all students are engaged in meaningful and thoughtful mathematical problem solving. To accomplish this goal, certain classroom norms must be established in which teachers and classmates learn to listen to the ideas of all students and to recognize, respect, and value their contributions (NCTM 2008).

We implemented such equitable practices during an informal mathematics learning program. Twenty-four African American and Latino student participants volunteered to work on open-ended mathematical tasks as an extra after-school activity. Our strategies significantly engaged them in justification and reasoning during problem solving.

We offer two representative episodes to show how students' reasoning was made public in justifying problem solutions as well as in responding to and challenging the ideas of others. The sixth graders' reasoning took the form of both direct and indirect proof.

The classroom community

We encouraged students to work together, share their ideas and conjectures, and listen to and question others' ideas. Seated in heterogeneous groups of four, students received a series of tasks—dealing with fraction ideas—for which they were to justify their solutions. For many of the students, the opportunity to work collaboratively on open-ended tasks was a new experience. Therefore, we chose tasks from our earlier research that had promoted collaborative reasoning and problem solving in the



past. **Table 1** outlines the tasks that we used during five ninety-minute sessions. Students were given a set of Cuisenaire® rods (see **fig. 1**) and were invited to build models of their solutions. The set contains ten colored wooden or plastic rods that increase in length by increments of one centimeter.

After a problem was posed to the class, students had the choice to work alone or collaboratively. During their initial exploration, students worked in pairs and groups, while the teacher moved from group to group and observed students' activity, listened to their ideas and explanations, and encouraged them to continue their investigation. As appropriate, students were invited to share ideas with group members or prepare solutions for group sharing. Teachers' questions were designed to better understand students' thinking, to encourage students to talk about their ideas and work, and sometimes to direct students' attention to an incomplete component of an argument or extend their investigation about a mathematical idea. Teachers encouraged students to broaden their knowledge about an approach to a solution by listening to one another and considering ideas from others (Mueller and Maher 2009).

The sixth graders represented their solutions in various ways, often building models that they used to explain their ideas. These models helped communicate alternative ways of representing solutions to others. Students were encouraged to listen to one another when they judged their peers' arguments, interjected their opinions, and offered alternate solutions. Teachers did not judge or evaluate students' ideas, solutions, and strategies, so students did not fear being wrong. They were asked to justify their reasoning to their classmates and assess whether their solutions were convincing. Hence, students took responsibility for posing questions about ideas and evaluating the reasonableness of arguments.

Over the course of the five sessions, students engaged in high-order reasoning that often led to justifications in the form of proof. To convince their peers of an argument's soundness, they used multiple representations and forms of reasoning (see **table 2**). The following episodes provide examples of students' reasoning.

TABLE 1

Students received these challenges during the first five sessions of their after-school program.

Date	Tasks
11-12-03	1. If I give the yellow rod the number name *five*, what number name would I give the orange rod? 2. Suppose I give the orange rod the number name *four*; what number name would I give the yellow rod? 3. If I call the orange rod *one*, what number name would I give the yellow rod? 4. If I call the white rod *two*, what number name would I give all the other rods?
11-13-03	1. Suppose I call the dark green rod *one*; what number name would I give the light green rod? 2. Someone told me that the red rod is half as long as the yellow rod; what do you think? 3. If I call the blue rod *one*, find me a rod that would have the number name *one-half*.
11-19-03	1. Convince us that there is not a rod that is half the length of the blue rod. 2. Is *0.3* another name for the light green rod? 3. If I call the blue rod *one*, what number name would I give the white rod? What name would I give the red rod?
11-20-03	1. If I call the blue rod *one*, what number name would I give the red rod? What name would I give the light green rod? 2. If I call the blue rod *one*, what number names would I give the rest of the rods?
12-03-03	1. If I call the orange rod *one*, what number name would I give the white rod? What name would I give the red rod? 2. If I call the orange rod *ten* (fifty), what number name would I give the white rod? 3. I want to know which is bigger, one-half or one-third, and by how much.

FIGURE 1

A set of Cuisenaire rods contains ten wooden or plastic rods of specific colors that increase in length by increments of one centimeter.

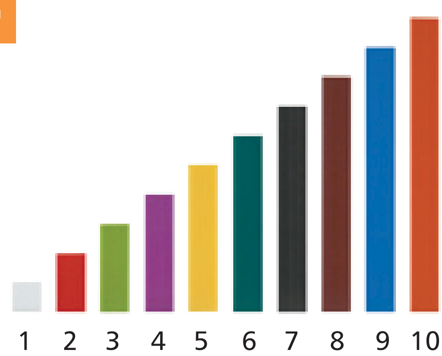


TABLE 2

Our study identified four forms of students' reasoning.

Form of Reasoning	Definition (for purposes of this study)
Direct	"A direct proof is based on the assumption that the hypothesis contains enough information to allow the construction of a series of logically connected steps leading to the conclusion" (Cupillari 2005, p. 12). Takes the form: $p \rightarrow q$
By Contradiction	Reasoning by contradiction, also known as the *indirect method*, is based on the agreement that whenever a statement is true, its contrapositive is also true or that a statement is equivalent to its contrapositive. For example, $p \rightarrow q$ is equivalent to (not q) $\rightarrow$ (not p); so if (not q) $\rightarrow$ (not p) is true, then $p \rightarrow q$ is also true (Cupillari 2005).
Using Upper and Lower Bounds	An *upper bound* of a subset *S* of some partially ordered set is an element that is greater than or equal to every element of *S*. The term *lower bound* of a subset *S* of some set refers to an element that is less than or equal to every element of *S*. An argument is then formed to justify a statement about the subset with the defined bounds (for example, that it is empty).
By Cases	For the purpose of this study, critical events were coded as *reasoning by cases* when students defended an argument by defending separate instances. Students defended an implication in the form $p \rightarrow q$, when p is a compound proposition composed of propositions p_1, p_2, … p_n, and they established each of the implications $p_1 \rightarrow q$ $p_2 \rightarrow q$, $p_n \rightarrow q$.

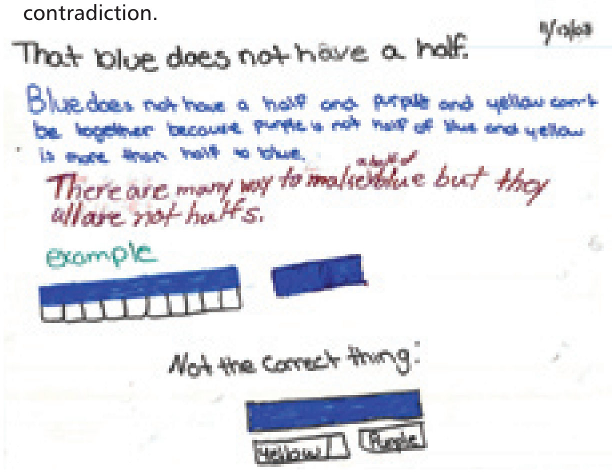
FIGURE 2

Small groups of students received a set of Cuisenaire rods and were invited to build models of their solutions.

(a) Shirelle and Michael erroneously called two rods of different sizes *one-half* the blue rod.

(b) Chris reasoned that the length of one blue rod equals nine white rods, which cannot be evenly divided in half.

(c) Chris defended his model with an argument by contradiction.

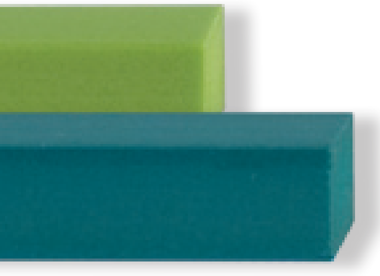


Episode 1

The problem was to find a rod with the number name *one-half* when the blue rod has the number name *one*. During the second session of the after-school math program, Shirelle and Michael proposed that the purple and yellow rods could serve as half the blue rod. They built a model showing the purple and yellow rods aligned with the blue rod and used direct reasoning to support their conjecture (see **fig. 2a**).

Dante challenged their claim, arguing that the purple and yellow rods are not the same length, so they could not be called *halves*.

Using a counter argument, Chanel explained, "And the yellow takes up more space than the purple; to be halves, they should be the same."



At another table, Chris proposed to his group that no rod in the set could be called *one-half*. He reasoned that nine white rods are equivalent to the length of the blue rod (see **fig. 2b**) and cannot be partitioned into two sets without a remainder. Chris defended his model using an argument by contradiction (see **fig. 2c**):

> Um, like y'all was saying, the white little rods won't be able to do it, but since there's nine white little rods, you can't really divide that into a half. So, you can't really divide by two because you get a decimal or remainder. So, there is really no half, no half of blue because of the white rods.

Also using a model (see **fig. 3**), Dante showed that the purple rod could not be considered half of the blue rod because the combination of two purple rods is not equivalent to the length of the blue rod (they are too short). Likewise, the yellow rod could not be named *half* of the blue rod because the combination of two yellow rods was not equivalent in length to the blue rod. Dante's argument uses upper and lower bounds.

Justina presented an alternative argument based on portioning the set of rods into two cases, those with halves and those without (*singles*), enumerating all the cases (see **fig. 4**).

Students used four different forms of reasoning to justify their solutions for this problem (see **table 2**). In only one case was the reasoning faulty (Michael and Shirelle's direct argument that the yellow and purple rods could both be called *half* the blue rod). As early as the second session, students listened to one another's ideas and proposed arguments for the reasonableness of their own solutions.

FIGURE 3

Dante offered an argument using upper and lower bounds.

FIGURE 4

Justina presented an argument based on portioning the set of rods into two cases: those with halves and those without.

Episode 2

During session 3, one student, Jeffrey, posed the task of naming the red rod when the blue rod is named *one*. At the beginning of session 4, students had an opportunity to build on their earlier problem solving. Individual groups of students initiated the challenge of naming all the rods (when the blue rod is named *one*).

At one table, Chanel named the remainder of the rods using direct reasoning based on the incremental increase by one white rod, or one-ninth. She used the staircase model (see **fig. 1**) as a guide and named the rods, increasing by one-ninth, hesitating at the orange rod, and then naming it *rod nine-tenths*. When she explained her dilemma—of naming the orange rod—to Dante, he initially named the orange rod *ten-ninths* but corrected himself and said that the orange rod would start a "new cycle" and be named *one-tenth*. Dante told the group that he heard students at other tables calling the orange rod *ten-ninths*. Michael insisted that the others were incorrect. Chanel agreed and claimed that "the denominator can't be smaller than the numerator." Dante concurred. They discussed a rule, which Michael referred to as *the laws of math* and *the laws of facts*, that states that the denominator cannot be smaller than the numerator.

Reminded by a teacher that the white rod is named *one-ninth*, Dante finally used the staircase model to name the orange rod *ten-ninths*. He explained that the length of ten white rods is equivalent to the length of an orange rod and that because a white rod is called *one-ninth*, the orange rod will be called *ten-ninths*. Later during the same session, students shared their findings with the whole class (see **table 3**). Five students presented arguments using direct reasoning. However, they based their arguments on different representations.

A culture of confidence

In a relatively short period of time, a culture of sense making, communication, and collaboration evolved over the five sessions. The first episode occurred during session 2 of the program. Students were already working together to build representations, questioning each other, and defending their solutions. Chanel and Dante collaborated in an attempt to convince Michael and Shirelle that the purple and yellow rods could not be called *one-half.* They built an argument using upper and lower bounds. During whole-class sharing, both students presented sophisticated versions of their argument.

The second episode began with students attending to the misconception that the numerator is always less than the denominator. By building a model, they convinced themselves and one another that the blue rod could indeed be named *ten-ninths*. Students were confident in sharing their justifications and secure with having representations that differed from their peers' models.

A culture of equity

Educators often suggest that minority-race, inner-city students "need structure." Adults therefore organize classrooms to focus on having students learn procedures and skills. This perspective leaves little room for students to reason, conjecture, and share ideas. As a result, students develop a view of mathematics as rule oriented and procedure driven (Powell 2004).

In contrast, our after-school informal math sessions focused on having students build personal meaning of mathematical ideas. As the above episodes illustrate, students actively engaged in solving problems and justifying their solutions. They posed arguments and defended those arguments. They questioned and corrected one another and ultimately created justifications that took the form of proofs. We documented four types of reasoning that students used to defend their arguments. Students used multiple representations to back up their claims and convince their classmates.

Putting dispositions into practice

Teachers can promote such sense making and reasoning by engaging students in similar activities in their own math classs. Give your students responsibility for justifying their problem-solving solutions. To encourage teachers, we share certain characteristics of the tasks and environment that led to successful problem solving:

1. Give choices. Seat students in small groups; participants then have options: to work

TABLE 3

Students were asked to share their direct arguments for naming the orange rod *ten-ninths* with the whole class.

Student	Direct Argument
Lorrin	Before, we thought that because we knew that the numerator would be larger than the denominator, and we thought that the denominator always had to be larger. But we found out that was not true because two yellow rods equal five-ninths and five-ninths plus five-ninths equal ten-ninths.
Kia-Lynn	The orange [rod] is bigger than the blue one, but when you add a one-ninth—a white rod—to the blue top, it kind of matches. We found out that you can also call the blue rod *one and one-ninth* and the orange *one* without the one-ninth; without the white rod is also called *one-ninth*, too. If you have one white rod and you add it to the blue, it's *one-ninth plus one* [or] *one and one-ninth*. So, if the blue rod and one white, if you put them together, then this means that it's *ten-ninths* also known as *one and one-ninth*.
Dante	Well, all I did was start from the beginning—start from the white—all the way to the orange and—like Kia-Lynn's group just said—I had found a different way to do it. Because all I had used [was] an orange, two purples, and a red. Since these two are purple, and this is supposed to be purple, but I had purple, and I used a red since four and four are eight, so, which will make it *eight-ninths* right here. And then plus two to make it *ten-ninths*. That's what I made.
Chanel	For all of these, I gave the white rod *one-ninth*, the red rod *two-ninths*, the light green rod *three-ninths*, *four-ninths* for the pink—uh, purple—rod, *five-ninths* for the yellow rod, *six-ninths* for the dark green rod, *seven-ninths* for the black rod, *eight-ninths* for the brown rod, *nine-ninths* for the blue rod, and for orange I gave it *one-ninth*.
Chris	I was saying *one and one-ninth* because if you add one blue one and since the number name for the blue is *one*, then if you add a white one, that equals *one-ninth*. If you add *one* and *one-ninth*, then that would equal *one and one-ninth*.

individually, with a partner, with a subset of the group, or with the entire group. Thus, students may engage in the way in which they learn best.

2. Differentiate. Appropriate teacher moves can facilitate how students share ideas. For example, to promote interest in the ideas of others, ask a student if he or she is aware of another student's solution. Point out different and contradictory claims and leave it to the students to work out the reasonableness of the arguments; this encourages them to share ideas and listen to one another. Allowing adequate time to explore, share, and revisit problems will respect the pace of slower-working students who may have different learning styles. Having extension tasks available is essential to challenging those who work more quickly.
3. Make ideas public. After students explore in their small groups, invite them to use an overhead projector to write ideas and share solutions. Have them make a variety of representations public and discuss them. Organize the order of the presentations by asking several students to share their solutions and strategies so that it becomes apparent that alternate paths and ways of representing mathematical ideas exist rather than only one "correct" way. Emphasize the importance of offering arguments that are convincing to classmates, not just to the teacher. When conflicts arise, ensure that ideas are public; a resolution can be postponed until convincing evidence is offered. Communicating their ideas will engage learners. Taking responsibility for explaining ideas can lead to increased student confidence and autonomy.
4. Select the best tasks and tools. Choose open-ended tasks that allow for multiple entry points at multiple levels. All students can work from their personal representation and form their own ideas from this starting point; thus, all students can realize success from the onset. Opt for tasks that are novel to the students so they do not have a strategy readily available. Then they must rely on their own (and their partners') resources to plan a strategy and build new knowledge.

   Revisit tasks or pose similar tasks so that students have time to reflect on their previous solutions and those of their classmates and incorporate these strategies into their solution, thus promoting more refined justifications.

   Have manipulative materials available and encourage students to build models to show their conjectures. Urge them to record their strategies with pictures, numbers, and words.
5. Hold high expectations. Students' mathematical development is crucially linked to our aspirations for them. We need strategies for putting our high expectations into practice. Helping students to develop a culture of reasoning supports them in meeting rigorous standards and working up to their mathematical potential.

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The mathematics learning program sessions were part of the Informal Mathematical Learning Project (IML) directed by Carolyn A. Maher, Arthur Powell, and Keith Weber, supported by a grant from the National Science Foundation (ROLE: REC0309062). For all sessions, the teacher or facilitator was a member of the research team. Tasks were chosen from previous research conducted under Grant MDR 9053597 from the National Science Foundation and Grant 93-992022-8001 from the New Jersey Department of Higher Education. Views expressed in this paper are those of the authors and not necessarily those of the funding agencies.

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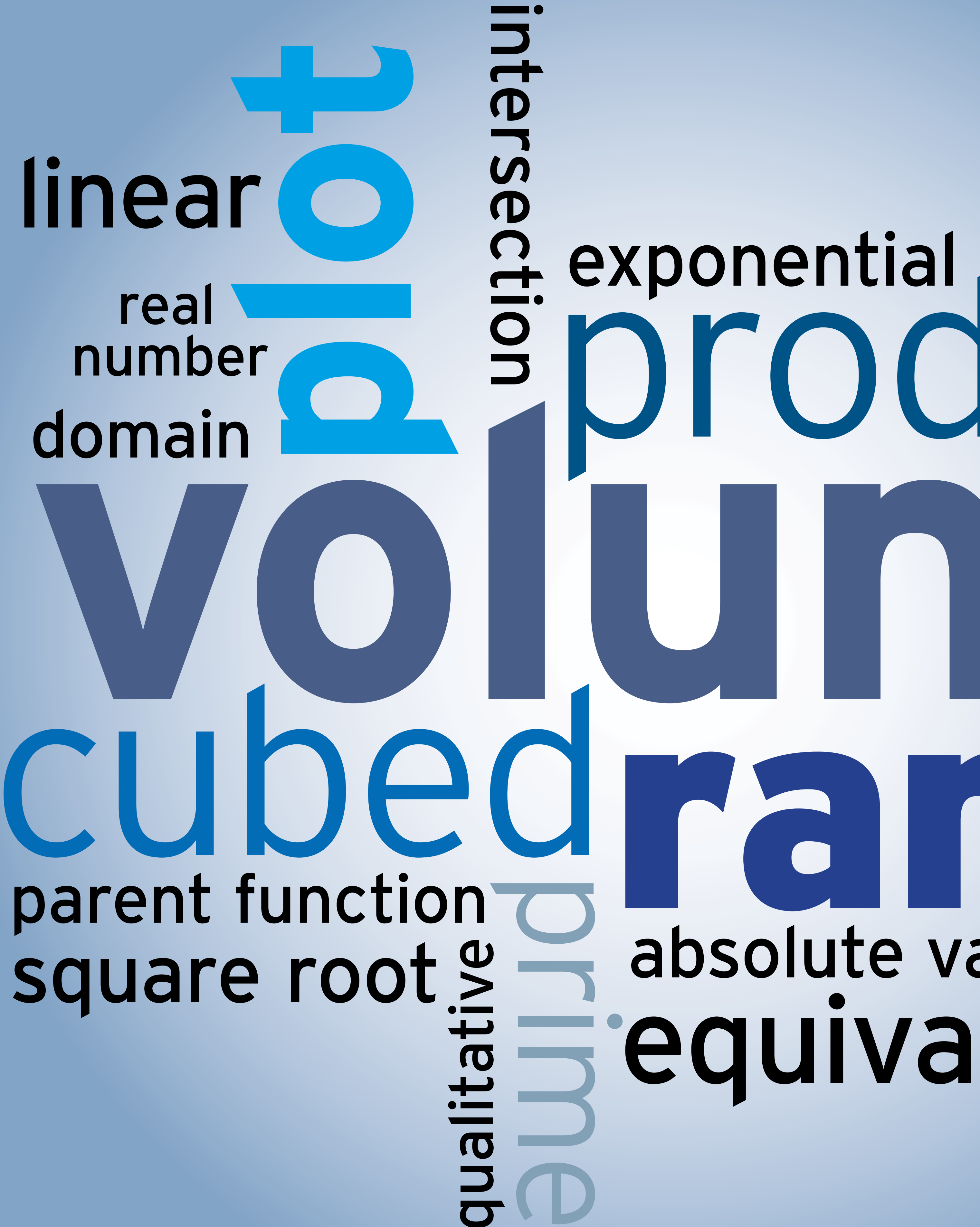
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For ELLs: Vocabulary beyond the Definitions

A classroom teacher discusses ambiguities in mathematics vocabulary and strategies for ELL students in building understanding.

Nancy S. Roberts and Mary P. Truxaw

"But where is the exponent?" Jorge, a tenth-grade English language learner (ELL), asked me while I (co-author Roberts) was talking about the formula for the area of a parallelogram. After much confusion on my part, Jorge said, "Last year you said that the base was the number in a power that was not the exponent. I don't see the exponent, so I don't know where the base is." Aha! I *had* said something like that in algebra class the previous year. However, I had never thought about the two different uses of the word *base* within mathematics.

Although I knew some of the challenges faced by ELL students learning mathematics vocabulary, I had never considered that mathematics, known for its precision, would include ambiguity *within* its vocabulary. In fact, the sixth Standard for Mathematical Practice within the Common Core State Standards for Mathematics (CCSSM) relates to attending to precision: "Mathematically proficient students" need to "communicate precisely to others" and "try to use clear definitions in discussion with others and in their own reasoning" (CCSSI 2010, p. 7).

I thought about Jorge. He had been confident enough and had the language skills to ask for clarification; many ELL students might not. If I had not recognized the connection to my earlier use of mathematics vocabulary, where would this confusion have led? How would I have uncovered it? How would Jorge's confusion have impeded mastering important mathematical practices or communicating precisely? These questions and others led to my investigation into the role of vocabulary development in helping ELL students be successful in mathematics, specifically in first-year algebra.

As I considered the importance of supporting ELL students' mathematics vocabulary, I asked myself a question that would likely arise for many mathematics teachers: "Do I have time to spend on vocabulary development?" Jorge helped me recognize that I had to ask myself a different question: "Can I afford *not* to spend time on vocabulary development?" Many vocabulary strategies that have worked for my students do not add much additional time and enhance not only vocabulary but also the mathematics.

CHALLENGES OF MATHEMATICS VOCABULARY FOR ELLS

Although mathematics language is much more than just learning vocabulary (Moschkovich 1999, 2002; NCTM 2000), vocabulary development is still central to learning to read, write, speak, listen to, and make sense of mathematics (CCSSI 2010; Heinze 2005). I will focus specifically on helping ELL students build better understanding of algebra through vocabulary, sharing outcomes of my own learning about mathematics vocabulary and strategies that worked for my students and me.

Mathematics vocabulary may be more difficult to learn than other academic vocabulary for several reasons.

Table 1 Math Usage vs. Everyday Usage

Vocabulary Word	Mathematics Usage	Everyday Usage
volume	Amount of space	Noise level
product	Result in multiplying	Item produced in manufacturing
plot	To graph a point	A piece of land to build a house
cubed	Raised to the third power	A type of steak or a way to cut vegetables
range	Numerical difference between two values	Stove top
prime	Prime number	Prime rib, prime time

Source: Adams (2003), p. 789

Table 2 Homonyms and Similar Sounding Words

whole – hole two – to – too tenths – tents	eight – ate symbol – cymbal half – have	sum – some sides – size real – reel

Source: Adams (2003), p. 790–91

1. Definitions are filled with technical vocabulary, symbols, and diagrams (Pimm 1987). Teachers need to explicitly help students make sense of this new language (Schlepegrell 2007).
2. Many mathematics concepts can be represented in multiple ways. At least thirteen different terms can mean subtraction (Echevarria, Vogt, and Short 2010; Heinze 2005). Multiplication can be indicated in many ways: "2 times 3," "2 multiplied by 3," and "the product of 2 and 3." To add to the confusion, some words may have similar connotations but vastly different technical meanings—for example, "3 multiplied by 10" and "3 increased by 10" (Heinze 2005).
3. Many mathematics words have multiple meanings. A *quarter* may refer to a coin or a fourth of a whole. Students must learn that the same word in different situations has different meanings, such as asking for a quarter while at a vending machine or while eating a pizza (Moschkovich 2002).
4. The overlap between mathematics vocabulary and everyday English (Kotsopoulos 2007; Moschkovich 2002) is problematic (see **table 1**). The word *product*, for instance, has meaning in everyday English that is completely different from its very specific mathematical meaning.
5. Homonyms and words that sound similar can confuse (Adams 2003). See **table 2** for a partial list.
6. Similarity to native language words can add more confusion. Although these similarities may sometimes be helpful—as when cognates have similar sounds and similar meanings—similarities can also contribute to confusion. For example, the Spanish word for *quarter* is *cuarto*, which can mean "a quarter of an hour"; *quarter* could also mean a room in a house, as in the English usage "your living quarters" (Moschkovich 1999, 2002).

Clearly, vocabulary is an important issue in mathematics classrooms, especially for ELL students.

TEACHING METHODS AND STRATEGIES

A selection of strategies for supporting students' development of mathematics vocabulary and examples of how to use them follow. Suggestions illustrate vocabulary support within an algebra unit but could be adapted for other topics. Two tools that will be highlighted are word walls—organized collections of words displayed in the classroom to support vocabulary development—and graphic organizers—visual charts and representations designed to organize student learning. We will also look at ways in which these tools can encompass vocabulary strategies.

Table 3 Sample Algebra Vocabulary					
absolute value	binomial	coefficient	complete the square	conjunction	derive
domain	equivalent	exponential	function	intersection	interval
inverse	linear	monomial	parabola	parent function	piecewise
polynomial	qualitative	quadratic	radicand	range	rational
real number	regression	solution	square root	trinomial	variable

Source: CCSSI (2010), pp. 52–71

Develop a Vocabulary List

Begin by developing a vocabulary list for the unit. **Table 3** shows samples of mathematics vocabulary from the Common Core State Standards for algebra (CCSSI 2010). Along with traditional algebra terms, include vocabulary to support challenges for ELLs, as described earlier (e.g., *symbol* and *whole*). Scaffolding such as word walls and graphic organizers will increase vocabulary usage while reducing cognitive load and stress (Echevarria, Vogt, and Short 2010).

Preteach and Assess

At the start of a unit, it is beneficial to trigger and assess prior knowledge, review previously learned vocabulary, and preteach new vocabulary. Preteaching vocabulary words requires explicit teaching of definitions, pronunciation, and word parts (Paulsen 2007).

Word Walls

One strategy for stimulating and assessing prior knowledge is a word wall. At the beginning of the unit, display all the vocabulary for the unit to act as an *anticipation guide*, a strategy used during preteaching to stimulate interest in a topic and give students a preview of what is to come. One way to use a word wall as a preassessment tool and as the trigger on the first day of a unit is to include a word that does not belong. Then ask small groups to pick out the word and describe why it does not belong. In a graphing unit, for example, the word wall could include the term *scientific notation* along with graphing words such as *slope*, *y-intercept*, *ordered pair*, *xy-intercepts*, and so on. (The nonconforming word would later be removed from the word wall.)

Another way to use word walls for preassessment is to have students organize the words into groups and give reasons for their choices. Words relating to a unit on exponents might be *base*, *exponent*, *denominator*, *numerator*, *polynomial*, *monomial*, *binomial*, *trinomial*, *power*, *reciprocal*, *coefficient*, and *factor*. One student might group *denominator*, *numerator*, and *reciprocal* as words related to fractions; another student might group *base*, *exponent*, and *power* as words describing a power. Listening to discussions provides interactive forms of preassessment. Moreover, student explanations provide opportunities to foster CCSS mathematical practices—for example, communicating precisely to others and constructing viable arguments.

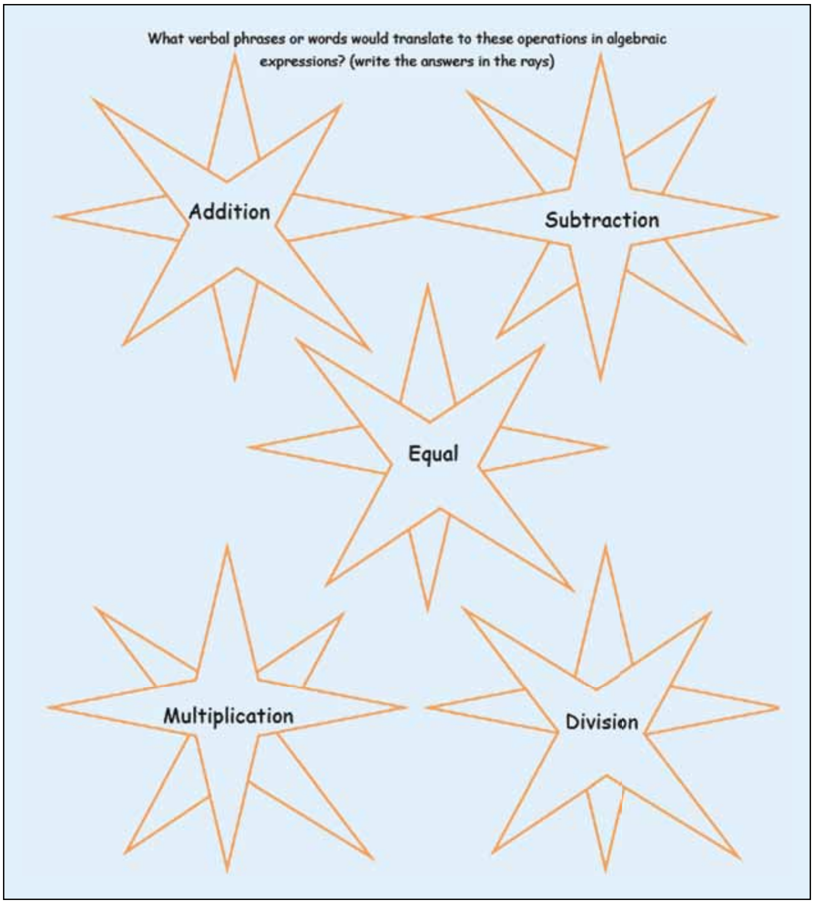

Fig. 1 The points of the stars provide space for students to write phrases that mean the same thing.

Graphic Organizers

Graphic organizers can be useful for activating and assessing students' prior knowledge, organizing different ways to express basic mathematical concepts, and organizing vocabulary for long-term retention. One organizer includes eight-sided stars with words for arithmetic operations and equality (see **fig. 1**). Working with partners, students list words that could be used for each operation. Then they add to their lists by comparing these in small groups. Finally, the class as a whole reviews the words. This class review is a time to make connections to the mathematical concepts, to address misconceptions, and to include words and phrases that are often confusing—for instance, "4 less than x" to mean "x minus 4."

Teach and Reteach

Researchers have provided many suggestions for explicitly teaching and reteaching vocabulary (see, e.g., Adams 2003; Gee 1996; Moschkovich 2002; Paulsen 2007). The focus here will be on word walls and graphic organizers.

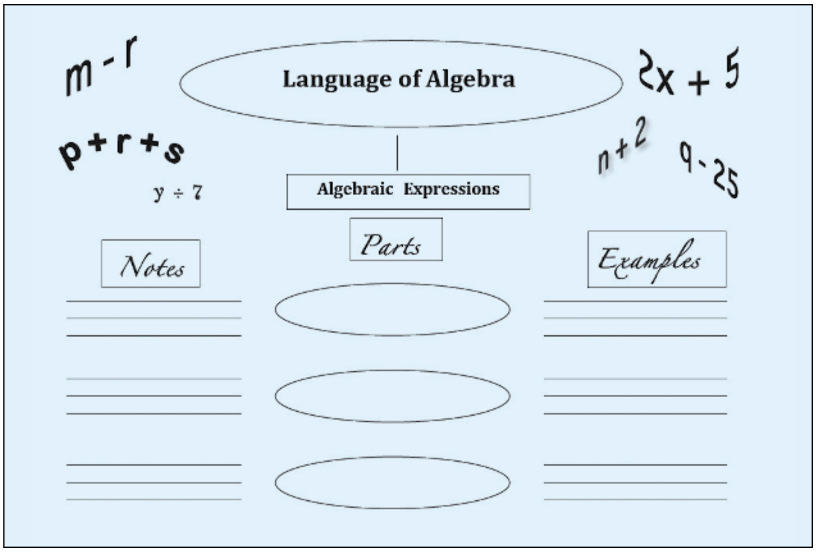

Fig. 2 *Variable*, *constant*, and *operation* would be appropriate entries in the ovals in the middle column.

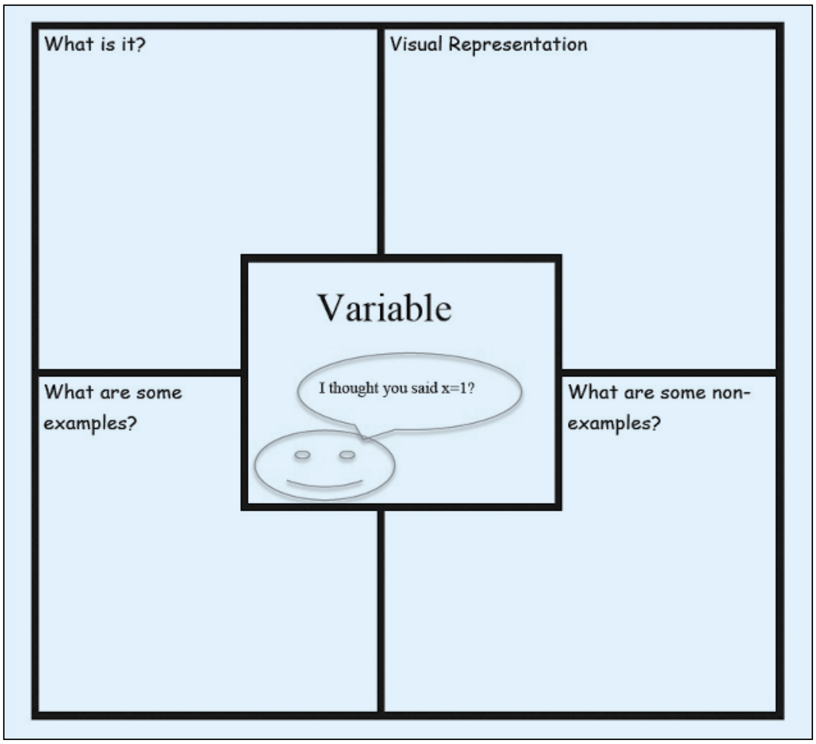

Fig. 3 A Frayer model is useful for some vocabulary.

Word Walls

Word walls are also useful within instructional units. A key idea is that word walls should be interactive, not static. After explicitly teaching words in the context of the unit, add definitions, examples, and diagrams to the words on the wall. Using nonexamples can help refine or clarify definitions (Adams 2003). In addition, real-life situations can provide context for algebra vocabulary and concepts (Paulsen 2007).

A helpful strategy is to start with informal definitions (while preteaching and assessing prior knowledge) and then transition to formal definitions (NCTM 2000). For example, the informal definition "a variable is a letter" may lead to "a variable is a symbol that represents a number" and finally to "a variable is a symbol, usually a letter, that is a quantity that can have different values." Informal definitions help students construct their own meaning, but formal definitions help them understand and apply concepts presented in mathematics textbooks (Adams 2003).

Ongoing interactive use of the word wall helps students see its value. As the year progresses, students use the word wall when answering verbal questions, when writing responses to essential questions on tests, and at other times when vocabulary usage is emphasized.

Graphic Organizers

Graphic organizers are beneficial within a unit of study to build and reinforce mathematics language. A graphic organizer entitled The Language of Algebra provides an opportunity to teach or reteach the parts of an algebraic expression by giving definitions and examples in the context of expressions (see **fig. 2**). In this specific organizer, the "parts" section (middle column) could list *variable*, *constant*, and *operation*, with notes and examples for each in the left and right columns. Similar language organizers could be developed for other topics.

A Frayer model is a specific graphic organizer that is useful when vocabulary terms are confusing or closely related (Barton and Heidema 2002). The model contains four sections: definition, facts, examples, and nonexamples (see **fig. 3** for an example related to the term *variable*). Both research (Adams 2003; Paulsen 2007) and personal experience demonstrate that nonexamples can be particularly powerful in helping refine and clarify definitions. When students ask, "How about this?" or "How about that?" they can refer to the example and nonexample sections. New misunderstandings that are uncovered can be added to the "nonexample" section. Sometimes substituting sections to suit the situation can be useful—for instance, using essential characteristics and nonessential characteristics or symbolic representation and graphical representation as sections. Students frequently refer to their organizers during lessons or when reviewing for tests.

Provide Repetition and Support Long-Term Retention

All students benefit from repeated exposure to vocabulary; however, ELLs require more repetition to integrate vocabulary into their mathematical understanding. In addition, students may need assistance in organizing their vocabulary knowledge into long-term memory (Adams 2003). Using vocabulary words within context while referring to the definitions (Echevarria, Vogt, and Short 2010) can be helpful. Providing different examples or diagrams each time the word is used helps avoid confusion and brings depth to students' growing understanding.

Word Walls

Reinforcing vocabulary from the interactive word wall can support long-term retention. A simple idea is to take four to five minutes at the end of class to play password or charades, using words from current or previous word walls. Another idea is to encourage and facilitate instructional conversations (Cazden 2001) that can support long-term retention of mathematics language and build meaning about mathematical concepts (NCTM 2000). Word walls can scaffold these conversations. When small groups discuss a mathematics problem, points can be awarded for appropriate use of words from the word wall—for example, using words such as *formula*, *variables*, *equations*, *graphs*, and *order of operations* when discussing using algebra in the real world.

Graphic Organizers

The graphic organizers used throughout a unit can and should be revisited to support long-term retention. In addition, new graphic organizers can be introduced to review previously learned vocabulary and concepts. For example, an organizer with a formal definition, specific properties or special cases, and some examples could be used to review the concept of factors (see **fig. 4**).

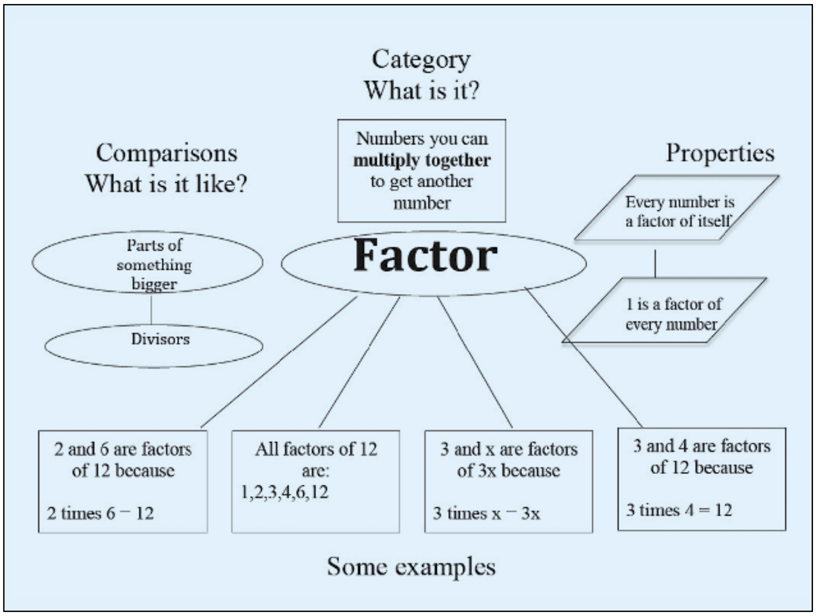

Fig. 4 This graphic organizer would be useful during review of the concept of factors.

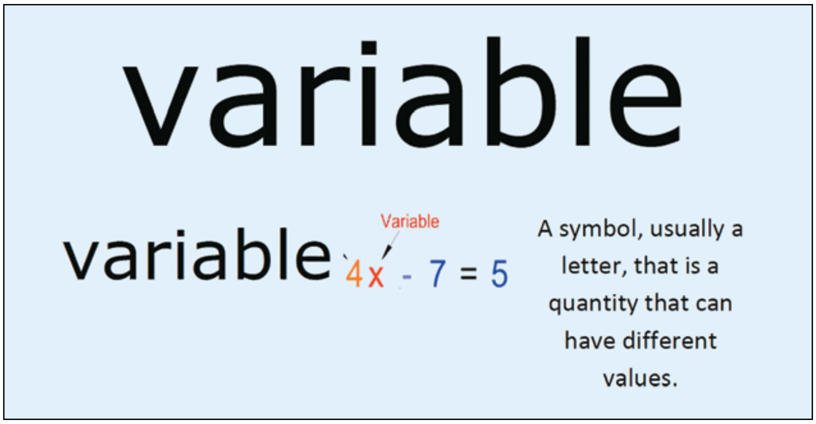

Fig. 5 This word wall entry can be folded so that only the vocabulary word is showing.

TEACHER AWARENESS

Along with reading research literature, mathematics teachers should build their own understanding of the challenges that their ELL students face. Awareness of the confusion caused by symbols and diagrams, concepts that can be represented with multiple terms, words that have multiple meanings, and the overlap between mathematics vocabulary and everyday usage can help teachers provide appropriate emphasis or explicit teaching.

HELPFUL HINTS

Word Walls

A simple way to make a word wall is to use a hanging pocket schedule organizer (typically used by elementary school teachers). After deciding on the unit vocabulary list (see **table 3**), type the words into a document (in landscape mode), with one word on each line. Center each word and enlarge it so that it fills a line of the paper. On the next line, type the word, its definition, a diagram, and an example. After printing, fold the paper so that the word is on one side and the expanded definition is on the other (see **fig. 5**). Slide the pieces into the organizer with the words showing. As the unit progresses and the words are discussed in context, reverse the paper so that the expanded definition is revealed.

Graphic Organizers

Many Internet sites—for example, CAST (www.cast.org) and Thinkport (www.thinkport.org)—have sample graphic organizers that can be used as is or customized. Teachers need not limit themselves to mathematics organizers; many excellent vocabulary organizers, such as Frayer models, come from other content areas. Providing a graphic organizer can help connect content within the unit and then can be used later as a review. Colored paper can assist with organization. In my class, colored paper means "keep it forever." Color makes important graphic organizers easy to find (I can say, "Pull out the red graphic organizer on variables"). At the end of the year, unit organizers make a good, concise way to review.

REFLECTIONS AND RECOMMENDATIONS

As I reflect on my experiences and those of my students, I am reminded of Jorge's confusion about mathematics vocabulary. His question has led me to increase my own awareness of the challenges related to mathematics vocabulary that ELL students face and strategies that I might use to support these students.

To help ELL students develop essential mathematical practices (CCSSI 2010), I recommend the use of word walls and graphic organizers to support vocabulary development. Specifically, I recommend the following:

- Select vocabulary words for a unit and post these on the day that the unit is introduced.
- Assess students' current understanding.
- Refer to the words throughout the unit, adding to the definitions and giving context.
- Provide frequent opportunities for students' misunderstanding to come to light.
- Use graphic organizers to help clarify the meaning of words and support long-term retention of vocabulary.

In addition to using word walls and graphic organizers, teachers should continue to investigate ideas available through books, journal articles, and websites (there are lots of good ideas out there). And, of course, listen to your students—that's the first step in supporting them.

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